

# COMPOSING GENERIC LINEARLY PERTURBED MAPPINGS AND IMMERSIONS/INJECTIONS

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**ABSTRACT.** Let  $f$  be an immersion of a manifold  $N$  into an open subspace  $U$  of  $\mathbb{R}^m$ . Let  $F : U \rightarrow \mathbb{R}^\ell$  be a mapping. Generally, the composition  $F \circ f$  does not necessarily yield a transverse mapping to a given subfiber-bundle of  $J^1(N, \mathbb{R}^\ell)$ . Nevertheless, in this paper, for any  $\mathcal{A}^1$ -invariant fiber, we show that composing generic linearly perturbed mappings of  $F$  and the given immersion  $f$  yields a transverse mapping to the subfiber-bundle of  $J^1(N, \mathbb{R}^\ell)$  with the given fiber. Moreover, we show a specialized transversality theorem on crossings of compositions of generic linearly perturbed mappings of a given mapping  $F : U \rightarrow \mathbb{R}^\ell$  and a given injection  $f : N \rightarrow U$ . Furthermore, applications of the two main theorems are given.

## 1. INTRODUCTION

Throughout this paper,  $\ell$ ,  $m$  and  $n$  stand for positive integers. In this paper, unless otherwise stated, all manifolds and mappings belong to class  $C^\infty$  and all manifolds are without boundary.

An  $n$ -dimensional manifold is denoted by  $N$ . Let  $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$  be a linear mapping. Let  $U$  be an open subspace of  $\mathbb{R}^m$ . Let  $F : U \rightarrow \mathbb{R}^\ell$  be a mapping. Then, set

$$F_\pi = F + \pi.$$

In the celebrated paper [10], by John Mather, for a given embedding  $f : N \rightarrow \mathbb{R}^m$ , the composition with the embedding  $\pi \circ f : N \rightarrow \mathbb{R}^\ell$  ( $m > \ell$ ) is investigated. The main theorem in [10] yields many applications. On the other hand, in this paper, for a given immersion (resp., injection)  $f : N \rightarrow U$ , the composition with the immersion (resp., injection)  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is investigated.

As some applications of two main theorems in this paper (Theorem 1 and Theorem 2 in Section 2), for a given immersion (resp., injection)  $f : N \rightarrow U$ , the following assertions (1)-(4) (resp., (5)) are obtained for a generic mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$ . Namely, their assertions are the properties obtained by generic linear perturbations of the given mapping  $F : U \rightarrow \mathbb{R}^\ell$ . The assertion (1)-(4) are obtained as applications of Theorem 1, and the assertion (5) is obtained as an application of Theorem 2.

- (1) If  $(n, \ell) = (n, 1)$ , then a generic function  $F_\pi \circ f : N \rightarrow \mathbb{R}$  is a Morse function.
- (2) If  $(n, \ell) = (n, 2n - 1)$  and  $n \geq 2$ , then any singular point of a generic mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^{2n-1}$  is a singular point of Whitney umbrella.

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2010 *Mathematics Subject Classification.* 57R35, 57R42, 57R45.

*Key words and phrases.* generic linear perturbation, transverse, immersion, injection, generalized distance-squared mapping .

Research Fellow DC1 of Japan Society for the Promotion of Science .

- (3) If  $\ell \geq 2n$ , then a generic mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is an immersion.
- (4) A generic mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  has corank at most  $k$  singular points, where  $k$  is the maximum integer satisfying  $(n - v + k)(\ell - v + k) \leq n$  ( $v = \min\{n, \ell\}$ ).
- (5) If  $\ell > 2n$ , then a generic mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is injective.

By combining the assertions (3) and (5), for a given embedding  $f : N \rightarrow U$ , the following assertion (6) is also obtained for a generic mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$ .

- (6) If  $\ell > 2n$  and  $N$  is compact, then a generic mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is an embedding.

Let  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  be the space consisting of linear mappings of  $\mathbb{R}^m$  into  $\mathbb{R}^\ell$ . For a given mapping  $f : N \rightarrow U$ , a property of mappings  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  will be said to be true for a *generic mapping* if there exists a subset  $\Sigma$  with Lebesgue measure zero of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , the mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  has the property.

For the proofs of the assertions (1)-(6), see Section 5.

**1.1. Remark.** In [4], for a given embedding  $f : N \rightarrow U$ , a generic mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is investigated. The important lemma for the proof of the main theorem in [4] is the main theorem in [10] (for the assertions of the main theorems in [10] and [4], see Theorem 3 and Theorem 4 in Section 7, respectively). On the other hand, the two main theorems (Theorem 1 and Theorem 2 in Section 2) in this paper are for compositions with a given immersion and a given injection, respectively. Moreover, they are shown without use of the main theorem in [4] and the main theorem in [10]. However, remark that the assertion (6) in Section 1, Corollary 7 in Section 5 and Corollary 9 in Section 6 are also corollaries obtained by using the main theorem in [4]. Since the paper [4] is under review at the time of submission of this paper, the assertion (6) in Section 1, Corollary 7 in Section 5 and Corollary 9 in Section 6 are written as corollaries obtained by the two main theorems in Section 2 of this paper.

For the sake of reader's convenience, the main theorem in [4] and the main theorem in [10] are introduced as an appendix (see Section 7). Remark that Section 7 is introduced only for Remark 1.1. Hence, Section 7 is not necessary except for Remark 1.1.

In Section 2, some of standard definitions are reviewed, and the two main theorems (Theorem 1 and Theorem 2) are stated. Section 3 (resp., Section 4) devotes the proof of Theorem 1 (resp., the proof of Theorem 2). In Section 5, as applications of the two main theorems, the assertions (1)-(6) are shown. Moreover, in Section 6, as further applications, the two main theorems are adapted to quadratic mappings of  $\mathbb{R}^m$  into  $\mathbb{R}^\ell$  of special type called "generalized distance-squared mappings" (for precise definition of generalized distance-squared mappings, see also Section 6). For the sake of reader's convenience, in Section 7, the main theorem in [4] and the main theorem in [10] are introduced for Remark in Subsection 1.1.

## 2. PRELIMINARIES AND THE STATEMENTS OF THE TWO MAIN THEOREMS

Let  $N$  and  $P$  be manifolds. Firstly, we recall the definition of transversality.

**Definition 1.** Let  $W$  be a submanifold of  $P$ . Let  $g : N \rightarrow P$  be a mapping.

- (1) We say that  $g : N \rightarrow P$  is *transverse* to  $W$  at  $q$  if  $g(q) \notin W$  or in the case of  $g(q) \in W$ , the following holds:

$$dg_q(T_q N) + T_{g(q)} W = T_{g(q)} P.$$

- (2) We say that  $g : N \rightarrow P$  is *transverse* to  $W$  if for any  $q \in N$ , the mapping  $g$  is transverse to  $W$  at  $q$ .

We say that  $g : N \rightarrow P$  is  $\mathcal{A}$ -*equivalent* to  $h : N \rightarrow P$  if there exist diffeomorphisms  $\Phi : N \rightarrow N$  and  $\Psi : P \rightarrow P$  such that  $g = \Psi \circ h \circ \Phi^{-1}$ .

Let  $J^r(N, P)$  be the space of  $r$ -jets of mappings of  $N$  into  $P$ . For a given mapping  $g : N \rightarrow P$ , the mapping  $j^r g : N \rightarrow J^r(N, P)$  is defined by  $q \mapsto j^r g(q)$  (for details on the space  $J^r(N, P)$  or the mapping  $j^r g : N \rightarrow J^r(N, P)$ , see for example, [3]).

For the statement and the proof of the first main theorem (Theorem 1), it is sufficient to consider the case of  $r = 1$  and  $P = \mathbb{R}^\ell$ . Let  $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$  be a coordinate neighborhood system of  $N$ . Let  $\Pi : J^1(N, \mathbb{R}^\ell) \rightarrow N \times \mathbb{R}^\ell$  be the natural projection defined by  $\Pi(j^1 g(q)) = (q, g(q))$ . Let  $\Phi_\lambda : \Pi^{-1}(U_\lambda \times \mathbb{R}^\ell) \rightarrow \varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times J^1(n, \ell)$  be the homeomorphism defined by

$$\Phi_\lambda(j^1 g(q)) = (\varphi_\lambda(q), g(q), j^1(g \circ \varphi_\lambda^{-1} \circ \tilde{\varphi}_\lambda)(0)),$$

where  $J^1(n, \ell) = \{j^1 g(0) \mid g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^\ell, 0)\}$  and  $\tilde{\varphi}_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the translation defined by  $\tilde{\varphi}_\lambda(0) = \varphi_\lambda(q)$ . Then,  $\{(\Pi^{-1}(U_\lambda \times \mathbb{R}^\ell), \Phi_\lambda)\}_{\lambda \in \Lambda}$  is a coordinate neighborhood system of  $J^1(N, \mathbb{R}^\ell)$ . A subset  $X$  of  $J^1(n, \ell)$  is called  $\mathcal{A}^1$ -*invariant* if for any  $j^1 g(0) \in X$ , and for any two germs of diffeomorphisms  $H : (\mathbb{R}^\ell, 0) \rightarrow (\mathbb{R}^\ell, 0)$  and  $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ , it follows that  $j^1(H \circ g \circ h^{-1})(0) \in X$ . Let  $X$  be an  $\mathcal{A}^1$ -invariant submanifold of  $J^1(n, \ell)$ . Set

$$X(N, \mathbb{R}^\ell) = \bigcup_{\lambda \in \Lambda} \Phi_\lambda^{-1}(\varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times X).$$

Then, the set  $X(N, \mathbb{R}^\ell)$  is a subfiber-bundle of  $J^1(N, \mathbb{R}^\ell)$  with the fiber  $X$  such that

$$\begin{aligned} \text{codim } X(N, \mathbb{R}^\ell) &= \dim J^1(N, \mathbb{R}^\ell) - \dim X(N, \mathbb{R}^\ell) \\ &= \dim J^1(n, \ell) - \dim X \\ &= \text{codim } X. \end{aligned}$$

Then, the first main theorem in this paper is the following.

**Theorem 1.** *Let  $N$  be a manifold of dimension  $n$ . Let  $f$  be an immersion of  $N$  into an open subspace  $U$  of  $\mathbb{R}^m$ . Let  $F : U \rightarrow \mathbb{R}^\ell$  be a mapping. If  $X$  is an  $\mathcal{A}^1$ -invariant submanifold of  $J^1(n, \ell)$ , then there exists a subset  $\Sigma$  with Lebesgue measure zero of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , the mapping  $j^1(F_\pi \circ f) : N \rightarrow J^1(N, \mathbb{R}^\ell)$  is transverse to the submanifold  $X(N, \mathbb{R}^\ell)$ .*

Now, in order to state the second main theorem (Theorem 2), we will prepare some definitions. Set  $N^{(s)} = \{(q_1, \dots, q_s) \in N^s \mid q_i \neq q_j \text{ (} i \neq j \text{)}\}$ . Notice that  $N^{(s)}$  is an open submanifold of  $N^s$ . Let  $g$  be a mapping of  $N$  into  $P$ . Then, let  $g^{(s)} : N^{(s)} \rightarrow P^s$  be the mapping defined by

$$g^{(s)}(q_1, \dots, q_s) = (g(q_1), \dots, g(q_s)).$$

Set  $\Delta_s = \{(y, \dots, y) \in P^s \mid y \in P\}$ . It is clearly seen that  $\Delta_s$  is a submanifold of  $P^s$  such that

$$\text{codim } \Delta_s = \dim P^s - \dim \Delta_s = (s - 1)\dim P.$$

**Definition 2.** Let  $g$  be a mapping of  $N$  into  $P$ . The mapping  $g$  is a *mapping with normal crossings* if for any positive integer  $s$  ( $s \geq 2$ ), the mapping  $g^{(s)} : N^{(s)} \rightarrow P^s$  is transverse to the submanifold  $\Delta_s$ .

For any injection  $f : N \rightarrow \mathbb{R}^m$ , set

$$s_f = \max \left\{ s \mid \forall (q_1, \dots, q_s) \in N^{(s)}, \dim \sum_{i=2}^s \overrightarrow{\mathbb{R}f(q_1)f(q_i)} = s - 1 \right\}.$$

Since the mapping  $f$  is injective, we get  $2 \leq s_f$ . Since  $f(q_1), \dots, f(q_{s_f})$  are  $s_f$ -points of  $\mathbb{R}^m$ , it follows that  $s_f \leq m + 1$ . Thus, remark that

$$2 \leq s_f \leq m + 1.$$

Then, the second main theorem in this paper is the following.

**Theorem 2.** Let  $N$  be a manifold of dimension  $n$ . Let  $f$  be an injection of  $N$  into an open subspace  $U$  of  $\mathbb{R}^m$ . Let  $F : U \rightarrow \mathbb{R}^\ell$  be a mapping. Then, there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , for any  $s$  ( $2 \leq s \leq s_f$ ), the mapping  $(F_\pi \circ f)^{(s)} : N^{(s)} \rightarrow (\mathbb{R}^\ell)^s$  is transverse to the submanifold  $\Delta_s$ . Moreover, if the mapping  $F_\pi$  satisfies that  $|F_\pi^{-1}(y)| \leq s_f$  for any  $y \in \mathbb{R}^\ell$ , then the mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is a mapping with normal crossings, where  $|X|$  is the number of elements of the set  $X$ .

The following well known result is important for the proofs of Theorem 1 and Theorem 2.

**Lemma 2.1** ([1], [10]). Let  $N, P, Z$  be manifolds, and let  $W$  be a submanifold of  $P$ . Let  $\Gamma : N \times Z \rightarrow P$  be a mapping. If  $\Gamma$  is transverse to  $W$ , then there exists a subset  $\Sigma$  of  $Z$  with Lebesgue measure zero such that for any  $p \in Z - \Sigma$ ,  $\Gamma_p : N \rightarrow P$  is transverse to  $W$ , where  $\Gamma_p(q) = \Gamma(q, p)$ .

### 2.1. Remark.

- (1) We explain the advantage that the domain of the mapping  $F$  is an arbitrary open set. Suppose that  $U = \mathbb{R}$ . Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be the mapping defined by  $x \mapsto |x|$ . Since  $F$  is not differentiable at  $x = 0$ , we can not apply Theorem 1 and Theorem 2 to the mapping  $F : \mathbb{R} \rightarrow \mathbb{R}$ .

On the other hand, if  $U = \mathbb{R} - \{0\}$ , then Theorem 1 and Theorem 2 can be applied to the restriction  $F|_U$ .

- (2) There is a case of  $s_f = 3$  as follows. If  $N = S^n$  and  $f : S^n \rightarrow \mathbb{R}^m$  is the inclusion  $f(x) = x$ , then it is clearly seen that  $s_f = 3$ . Indeed, suppose that there exists a point  $(q_1, q_2, q_3) \in (S^n)^{(3)}$  such that  $\dim \sum_{i=2}^3 \overrightarrow{\mathbb{R}f(q_1)f(q_i)} = 1$ . Then, since the number of the intersections of  $f(S^n)$  ( $= S^n$ ) and a straight line of  $\mathbb{R}^m$  is two and less, this contradicts the assumption. Thus, we get  $s_f \geq 3$ . By  $S^1 \times \{0\} \subset S^n$ , it follows that  $s_f < 4$ . Hence, we have  $s_f = 3$ .
- (3) The essential idea for the proofs of the two main theorems (Theorem 1 and Theorem 2) in this paper is to apply Lemma 2.1, and it is almost similar to the idea of the proofs of main results in [8]. Nevertheless, the two main theorems in this paper are drastically improved. As an effect of the improvement, many applications are obtained by the two main theorems (for the applications, see Section 5 and Section 6).

## 3. PROOF OF THEOREM 1

Let  $(\alpha_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$  be a representing matrix of a linear mapping  $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$ . Set  $F_\alpha = F_\pi$ , and we have

$$F_\alpha(x) = \left( F_1(x) + \sum_{j=1}^m \alpha_{1j} x_j, \dots, F_\ell(x) + \sum_{j=1}^m \alpha_{\ell j} x_j \right),$$

where  $F = (F_1, \dots, F_\ell)$ ,  $\alpha = (\alpha_{11}, \dots, \alpha_{1m}, \dots, \alpha_{\ell 1}, \dots, \alpha_{\ell m}) \in (\mathbb{R}^m)^\ell$  and  $x = (x_1, \dots, x_m)$ . For the given immersion  $f : N \rightarrow U$ , a mapping  $F_\alpha \circ f : N \rightarrow \mathbb{R}^\ell$  is as follows:

$$F_\alpha \circ f = \left( F_1 \circ f + \sum_{j=1}^m \alpha_{1j} f_j, \dots, F_\ell \circ f + \sum_{j=1}^m \alpha_{\ell j} f_j \right),$$

where  $f = (f_1, \dots, f_m)$ . Since there is the natural identification  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$ , in order to prove Theorem 1, it is sufficient to show that there exists a subset  $\Sigma$  with Lebesgue measure zero of  $(\mathbb{R}^m)^\ell$  such that for any  $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$ , the mapping  $j^1(F_\alpha \circ f) : N \rightarrow J^1(N, \mathbb{R}^\ell)$  is transverse to the give submanifold  $X(N, \mathbb{R}^\ell)$ .

Now, let  $\Gamma : N \times (\mathbb{R}^m)^\ell \rightarrow J^1(N, \mathbb{R}^\ell)$  be the mapping defined by

$$\Gamma(q, \alpha) = j^1(F_\alpha \circ f)(q).$$

If it follows that the mapping  $\Gamma$  is transverse to the submanifold  $X(N, \mathbb{R}^\ell)$ , then by Lemma 2.1, it follows that there exists a subset  $\Sigma$  of  $(\mathbb{R}^m)^\ell$  with Lebesgue measure zero such that for any  $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$ , the mapping  $\Gamma_\alpha : N \rightarrow J^1(N, \mathbb{R}^\ell)$  ( $\Gamma_\alpha = j^1(F_\alpha \circ f)$ ) is transverse to the submanifold  $X(N, \mathbb{R}^\ell)$ . Thus, in order to finish the proof of Theorem 1, it is sufficient to show that if  $\Gamma(\tilde{q}, \tilde{\alpha}) \in X(N, \mathbb{R}^\ell)$ , then the following (\*) holds.

$$(*) \quad d\Gamma_{(\tilde{q}, \tilde{\alpha})}(T_{(\tilde{q}, \tilde{\alpha})}(N \times (\mathbb{R}^m)^\ell)) + T_{\Gamma(\tilde{q}, \tilde{\alpha})}X(N, \mathbb{R}^\ell) = T_{\Gamma(\tilde{q}, \tilde{\alpha})}J^1(N, \mathbb{R}^\ell).$$

As per Section 2, let  $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$  (resp.,  $\{(\Pi^{-1}(U_\lambda \times \mathbb{R}^\ell), \Phi_\lambda)\}_{\lambda \in \Lambda}$ ) be a coordinate neighborhood system of  $N$  (resp.,  $J^1(N, \mathbb{R}^\ell)$ ). There exists a coordinate neighborhood  $(U_{\tilde{\lambda}} \times (\mathbb{R}^m)^\ell, \varphi_{\tilde{\lambda}} \times id)$  containing the point  $(\tilde{q}, \tilde{\alpha})$  of  $N \times (\mathbb{R}^m)^\ell$ , where  $id$  is the identity mapping of  $(\mathbb{R}^m)^\ell$  into  $(\mathbb{R}^m)^\ell$ , and the mapping  $\varphi_{\tilde{\lambda}} \times id : U_{\tilde{\lambda}} \times (\mathbb{R}^m)^\ell \rightarrow \varphi_{\tilde{\lambda}}(U_{\tilde{\lambda}}) \times (\mathbb{R}^m)^\ell \subset \mathbb{R}^n \times (\mathbb{R}^m)^\ell$  is defined by  $(\varphi_{\tilde{\lambda}} \times id)(q, \alpha) = (\varphi_{\tilde{\lambda}}(q), id(\alpha))$ . There exists a coordinate neighborhood  $(\Pi^{-1}(U_{\tilde{\lambda}} \times \mathbb{R}^\ell), \Phi_{\tilde{\lambda}})$  containing the point  $\Gamma(\tilde{q}, \tilde{\alpha})$  of  $J^1(N, \mathbb{R}^\ell)$ . Let  $t = (t_1, \dots, t_n) \in \mathbb{R}^n$  be a local coordinate on  $\varphi_{\tilde{\lambda}}(U_{\tilde{\lambda}})$

containing  $\varphi_{\tilde{\lambda}}(\tilde{q})$ . Then, the mapping  $\Gamma$  is locally given by the following:

$$\begin{aligned}
& (\Phi_{\tilde{\lambda}} \circ \Gamma \circ (\varphi_{\tilde{\lambda}} \times id)^{-1})(t, \alpha) \\
&= (\Phi_{\tilde{\lambda}} \circ \Gamma \circ (\varphi_{\tilde{\lambda}}^{-1} \times id^{-1}))(t, \alpha) \\
&= (\Phi_{\tilde{\lambda}} \circ \Gamma)(\varphi_{\tilde{\lambda}}^{-1}(t), \alpha) \\
&= \Phi_{\tilde{\lambda}}(\Gamma(\varphi_{\tilde{\lambda}}^{-1}(t), \alpha)) \\
&= \Phi_{\tilde{\lambda}}(j^1(F_{\alpha} \circ f)(\varphi_{\tilde{\lambda}}^{-1}(t))) \\
&= (\Phi_{\tilde{\lambda}} \circ j^1(F_{\alpha} \circ f) \circ \varphi_{\tilde{\lambda}}^{-1})(t) \\
&= \left( t, (F_{\alpha} \circ f \circ \varphi_{\tilde{\lambda}}^{-1})(t), \right. \\
&\quad \left. \frac{\partial(F_{\alpha,1} \circ f \circ \varphi_{\tilde{\lambda}}^{-1})}{\partial t_1}(t), \dots, \frac{\partial(F_{\alpha,1} \circ f \circ \varphi_{\tilde{\lambda}}^{-1})}{\partial t_n}(t), \right. \\
&\quad \left. \dots, \dots, \right. \\
&\quad \left. \frac{\partial(F_{\alpha,\ell} \circ f \circ \varphi_{\tilde{\lambda}}^{-1})}{\partial t_1}(t), \dots, \frac{\partial(F_{\alpha,\ell} \circ f \circ \varphi_{\tilde{\lambda}}^{-1})}{\partial t_n}(t) \right) \\
&= \left( t, (F_{\alpha} \circ f \circ \varphi_{\tilde{\lambda}}^{-1})(t), \right. \\
&\quad \frac{\partial F_1 \circ \tilde{f}}{\partial t_1}(t) + \sum_{j=1}^m \alpha_{1j} \frac{\partial \tilde{f}_j}{\partial t_1}(t), \dots, \frac{\partial F_1 \circ \tilde{f}}{\partial t_n}(t) + \sum_{j=1}^m \alpha_{1j} \frac{\partial \tilde{f}_j}{\partial t_n}(t), \\
&\quad \dots, \dots, \\
&\quad \left. \frac{\partial F_{\ell} \circ \tilde{f}}{\partial t_1}(t) + \sum_{j=1}^m \alpha_{\ell j} \frac{\partial \tilde{f}_j}{\partial t_1}(t), \dots, \frac{\partial F_{\ell} \circ \tilde{f}}{\partial t_n}(t) + \sum_{j=1}^m \alpha_{\ell j} \frac{\partial \tilde{f}_j}{\partial t_n}(t) \right),
\end{aligned}$$

where  $F_{\alpha} = (F_{\alpha,1}, \dots, F_{\alpha,\ell})$  and  $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_m) = (f_1 \circ \varphi_{\tilde{\lambda}}^{-1}, \dots, f_m \circ \varphi_{\tilde{\lambda}}^{-1}) = f \circ \varphi_{\tilde{\lambda}}^{-1}$ . The Jacobian matrix of the mapping  $\Gamma$  at  $(\tilde{q}, \tilde{\alpha})$  is the following:

$$J\Gamma_{(\tilde{q}, \tilde{\alpha})} = \left( \begin{array}{c|cccc} E_n & 0 & \cdots & \cdots & 0 \\ \hline & * & \cdots & \cdots & * \\ & {}^t(Jf_{\tilde{q}}) & & 0 & \\ * & & {}^t(Jf_{\tilde{q}}) & & \\ & & 0 & \ddots & \\ & & & & {}^t(Jf_{\tilde{q}}) \end{array} \right)_{(t, \alpha) = (\varphi_{\tilde{\lambda}}(\tilde{q}), \tilde{\alpha})},$$

where  $E_n$  is the  $n \times n$  unit matrix and  $Jf_{\tilde{q}}$  is the Jacobian matrix of the mapping  $f$  at  $\tilde{q}$ . Remark that  ${}^tA$  means the transposed matrix of  $A$ . Since  $X(N, \mathbb{R}^{\ell})$  is a subfiber-bundle of  $J^1(N, \mathbb{R}^{\ell})$  with the fiber  $X$ , in order to show (\*), it is clearly seen that the rank of the following matrix  $M_1$  is  $n + \ell + n\ell$ .

$$M_1 = \left( \begin{array}{c|cccc} E_{n+\ell} & * & \cdots & \cdots & * \\ \hline & {}^t(Jf_{\tilde{q}}) & & 0 & \\ 0 & & {}^t(Jf_{\tilde{q}}) & & \\ & & 0 & \ddots & \\ & & & & {}^t(Jf_{\tilde{q}}) \end{array} \right)_{(t, \alpha) = (\varphi_{\tilde{\lambda}}(\tilde{q}), \tilde{\alpha})},$$

where  $E_{n+\ell}$  is the  $(n+\ell) \times (n+\ell)$  unit matrix. Notice that for any  $i$  ( $1 \leq i \leq m\ell$ ), the  $(n+\ell+i)$ -th column vector of  $M_1$  is the  $(n+i)$ -th column vector of  $J\Gamma_{(\tilde{q}, \tilde{\alpha})}$ . Since the mapping  $f$  is an immersion ( $n \leq m$ ), we have that the rank of the matrix  $M_1$  is  $n+\ell+n\ell$ . Hence, we have (\*).

□

#### 4. PROOF OF THEOREM 2

Let  $(\alpha_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$  be a representing matrix of a linear mapping  $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$ . Set  $F_\alpha = F_\pi$ , and we have

$$F_\alpha(x) = \left( F_1(x) + \sum_{j=1}^m \alpha_{1j}x_j, \dots, F_\ell(x) + \sum_{j=1}^m \alpha_{\ell j}x_j \right),$$

where  $F = (F_1, \dots, F_\ell)$ ,  $\alpha = (\alpha_{11}, \dots, \alpha_{1m}, \dots, \alpha_{\ell 1}, \dots, \alpha_{\ell m}) \in (\mathbb{R}^m)^\ell$  and  $x = (x_1, \dots, x_m)$ . For the given injection  $f : N \rightarrow U$ , a mapping  $F_\alpha \circ f : N \rightarrow \mathbb{R}^\ell$  is as follows:

$$F_\alpha \circ f = \left( F_1 \circ f + \sum_{j=1}^m \alpha_{1j}f_j, \dots, F_\ell \circ f + \sum_{j=1}^m \alpha_{\ell j}f_j \right),$$

where  $f = (f_1, \dots, f_m)$ . Since there is the natural identification  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$ , in order to show that there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , for any  $s$  ( $2 \leq s \leq s_f$ ), the mapping  $(F_\pi \circ f)^{(s)} : N^{(s)} \rightarrow (\mathbb{R}^\ell)^s$  is transverse to the submanifold  $\Delta_s$ , it is sufficient to show that there exists a subset  $\Sigma$  of  $(\mathbb{R}^m)^\ell$  with Lebesgue measure zero such that for any  $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$ , for any  $s$  ( $2 \leq s \leq s_f$ ), the mapping  $(F_\alpha \circ f)^{(s)} : N^{(s)} \rightarrow (\mathbb{R}^\ell)^s$  is transverse to  $\Delta_s$ .

Now, let  $s$  be a positive integer satisfying  $2 \leq s \leq s_f$ . Let  $\Gamma : N^{(s)} \times (\mathbb{R}^m)^\ell \rightarrow (\mathbb{R}^\ell)^s$  be the mapping defined by

$$\Gamma(q_1, \dots, q_s, \alpha) = ((F_\alpha \circ f)(q_1), \dots, (F_\alpha \circ f)(q_s)).$$

If it follows that for any positive integer  $s$  ( $2 \leq s \leq s_f$ ), the mapping  $\Gamma$  is transverse to  $\Delta_s$ , then by Lemma 2.1, it follows that for any positive integer  $s$  ( $2 \leq s \leq s_f$ ), there exists a subset  $\Sigma_s$  of  $(\mathbb{R}^m)^\ell$  with Lebesgue measure zero such that for any  $\alpha \in (\mathbb{R}^m)^\ell - \Sigma_s$ , the mapping  $\Gamma_\alpha : N^{(s)} \rightarrow (\mathbb{R}^\ell)^s$  ( $\Gamma_\alpha = (F_\alpha \circ f)^{(s)}$ ) is transverse to  $\Delta_s$ . Then, set  $\Sigma = \bigcup_{i=2}^{s_f} \Sigma_i$ . It is clearly seen that  $\Sigma$  is a subset of  $(\mathbb{R}^m)^\ell$  with Lebesgue measure zero. Therefore, it follows that for any  $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$ , for any  $s$  ( $2 \leq s \leq s_f$ ), the mapping  $\Gamma_\alpha : N^{(s)} \rightarrow (\mathbb{R}^\ell)^s$  ( $\Gamma_\alpha = (F_\alpha \circ f)^{(s)}$ ) is transverse to  $\Delta_s$ .

Hence, in order to show that there exists a subset  $\Sigma$  of  $(\mathbb{R}^m)^\ell$  with Lebesgue measure zero such that for any  $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$ , for any  $s$  ( $2 \leq s \leq s_f$ ), the mapping  $(F_\alpha \circ f)^{(s)} : N^{(s)} \rightarrow (\mathbb{R}^\ell)^s$  is transverse to  $\Delta_s$ , it is sufficient to show that for any positive integer  $s$  ( $2 \leq s \leq s_f$ ), if  $\Gamma(\tilde{q}, \tilde{\alpha}) \in \Delta_s$  ( $\tilde{q} = (\tilde{q}_1, \dots, \tilde{q}_s)$ ), then the following (\*\*) holds.

$$(**) \quad d\Gamma_{(\tilde{q}, \tilde{\alpha})}(T_{(\tilde{q}, \tilde{\alpha})}(N^{(s)} \times (\mathbb{R}^m)^\ell)) + T_{\Gamma(\tilde{q}, \tilde{\alpha})}\Delta_s = T_{\Gamma(\tilde{q}, \tilde{\alpha})}(\mathbb{R}^\ell)^s.$$

Let  $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$  be a coordinate neighborhood system of  $N$ . There exists a coordinate neighborhood  $(U_{\tilde{\lambda}_1} \times \dots \times U_{\tilde{\lambda}_s} \times (\mathbb{R}^m)^\ell, \varphi_{\tilde{\lambda}_1} \times \dots \times \varphi_{\tilde{\lambda}_s} \times id)$  containing the point  $(\tilde{q}, \tilde{\alpha})$  of  $N^{(s)} \times (\mathbb{R}^m)^\ell$ , where  $id$  is the identity mapping of  $(\mathbb{R}^m)^\ell$  into  $(\mathbb{R}^m)^\ell$ , and the mapping  $\varphi_{\tilde{\lambda}_1} \times \dots \times \varphi_{\tilde{\lambda}_s} \times id : U_{\tilde{\lambda}_1} \times \dots \times U_{\tilde{\lambda}_s} \times (\mathbb{R}^m)^\ell \rightarrow (\mathbb{R}^n)^s \times (\mathbb{R}^m)^\ell$

is defined by  $(\varphi_{\tilde{\lambda}_1} \times \cdots \times \varphi_{\tilde{\lambda}_s} \times id)(q_1, \dots, q_s, \alpha) = (\varphi_{\tilde{\lambda}_1}(q_1), \dots, \varphi_{\tilde{\lambda}_s}(q_s), id(\alpha))$ . Let  $t_i = (t_{i1}, \dots, t_{in})$  be a local coordinate containing  $\varphi_{\tilde{\lambda}_i}(\tilde{q}_i)$  ( $1 \leq i \leq s$ ). Then, the mapping  $\Gamma$  is locally given by the following:

$$\begin{aligned}
& \Gamma \circ \left( \varphi_{\tilde{\lambda}_1} \times \cdots \times \varphi_{\tilde{\lambda}_s} \times id \right)^{-1} (t_1, \dots, t_s, \alpha) \\
&= \Gamma \circ \left( \varphi_{\tilde{\lambda}_1}^{-1} \times \cdots \times \varphi_{\tilde{\lambda}_s}^{-1} \times id^{-1} \right) (t_1, \dots, t_s, \alpha) \\
&= \Gamma \left( \varphi_{\tilde{\lambda}_1}^{-1}(t_1), \dots, \varphi_{\tilde{\lambda}_s}^{-1}(t_s), \alpha \right) \\
&= \left( (F_\alpha \circ f \circ \varphi_{\tilde{\lambda}_1}^{-1})(t_1), \dots, (F_\alpha \circ f \circ \varphi_{\tilde{\lambda}_s}^{-1})(t_s) \right) \\
&= \left( F_1 \circ \tilde{f}(t_1) + \sum_{j=1}^m \alpha_{1j} \tilde{f}_j(t_1), \dots, F_\ell \circ \tilde{f}(t_1) + \sum_{j=1}^m \alpha_{\ell j} \tilde{f}_j(t_1), \right. \\
&\quad \dots, \dots, \\
&\quad \left. F_1 \circ \tilde{f}(t_s) + \sum_{j=1}^m \alpha_{1j} \tilde{f}_j(t_s), \dots, F_\ell \circ \tilde{f}(t_s) + \sum_{j=1}^m \alpha_{\ell j} \tilde{f}_j(t_s) \right),
\end{aligned}$$

where  $\tilde{f}(t_i) = (\tilde{f}_1(t_i), \dots, \tilde{f}_m(t_i)) = (f_1 \circ \varphi_{\tilde{\lambda}_i}^{-1}(t_i), \dots, f_m \circ \varphi_{\tilde{\lambda}_i}^{-1}(t_i))$  ( $1 \leq i \leq s$ ). For simplicity, set  $t = (t_1, \dots, t_s)$  and  $z = (\varphi_{\tilde{\lambda}_1} \times \cdots \times \varphi_{\tilde{\lambda}_s})(\tilde{q}_1, \dots, \tilde{q}_s)$ .

The Jacobian matrix of the mapping  $\Gamma$  at  $(\tilde{q}, \tilde{\alpha})$  is the following:

$$J\Gamma_{(\tilde{q}, \tilde{\alpha})} = \left( \begin{array}{c|cccc} & \mathbf{b}(t_1) & & & 0 \\ & & \mathbf{b}(t_1) & & \\ & 0 & & \ddots & \\ & \mathbf{b}(t_2) & & & \mathbf{b}(t_1) \\ & & \mathbf{b}(t_2) & & 0 \\ * & 0 & & \ddots & \\ & & & & \mathbf{b}(t_2) \\ & \vdots & \vdots & \vdots & \vdots \\ & \vdots & \vdots & \vdots & \vdots \\ & \mathbf{b}(t_s) & & & 0 \\ & & \mathbf{b}(t_s) & & \\ & 0 & & \ddots & \\ & & & & \mathbf{b}(t_s) \end{array} \right)_{(t, \alpha) = (z, \tilde{\alpha})},$$



where  $\mathbf{b}(t_i) = (\tilde{f}_1(t_i), \dots, \tilde{f}_m(t_i))$ . By seeing the construction of  $T_{\Gamma(\tilde{q}, \tilde{\alpha})}\Delta_s$ , in order to show (\*\*), it is sufficient to show that the rank of the following matrix  $M_2$  is  $\ell s$ .

$$M_2 = \left( \begin{array}{c|cccc} E_\ell & \mathbf{b}(t_1) & & & 0 \\ & & \mathbf{b}(t_1) & & \\ & 0 & & \ddots & \\ & & & & \mathbf{b}(t_1) \\ \hline E_\ell & \mathbf{b}(t_2) & & & 0 \\ & & \mathbf{b}(t_2) & & \\ & 0 & & \ddots & \\ & & & & \mathbf{b}(t_2) \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline E_\ell & \mathbf{b}(t_s) & & & 0 \\ & & \mathbf{b}(t_s) & & \\ & 0 & & \ddots & \\ & & & & \mathbf{b}(t_s) \end{array} \right)_{t=z}.$$

There exists an  $\ell s \times \ell s$  regular matrix  $Q_1$  such that

$$Q_1 M_2 = \left( \begin{array}{c|cccc} E_\ell & \mathbf{b}(t_1) & & & 0 \\ & & \mathbf{b}(t_1) & & \\ & 0 & & \ddots & \\ & & & & \mathbf{b}(t_1) \\ \hline 0 & \mathbf{b}(t_2) - \mathbf{b}(t_1) & & & 0 \\ & & \mathbf{b}(t_2) - \mathbf{b}(t_1) & & \\ & 0 & & \ddots & \\ & & & & \mathbf{b}(t_2) - \mathbf{b}(t_1) \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & \mathbf{b}(t_s) - \mathbf{b}(t_1) & & & 0 \\ & & \mathbf{b}(t_s) - \mathbf{b}(t_1) & & \\ & 0 & & \ddots & \\ & & & & \mathbf{b}(t_s) - \mathbf{b}(t_1) \end{array} \right)_{t=z}.$$

Namely,

$$Q_1 M_2 = \left( \begin{array}{c|cccc} E_\ell & \mathbf{b}(t_1) & & & 0 \\ & 0 & & \ddots & \\ & & \mathbf{b}(t_1) & & \\ & & & \ddots & \\ & & & & \mathbf{b}(t_1) \\ \hline 0 & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_2)} & & & 0 \\ & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_2)} & & \\ & 0 & & \ddots & \\ & & & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_2)} \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_s)} & & & 0 \\ & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_s)} & & \\ & 0 & & \ddots & \\ & & & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_s)} \end{array} \right)_{t=z},$$

where  $\overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_i)} = (\widetilde{f}_1(t_i) - \widetilde{f}_1(t_1), \dots, \widetilde{f}_m(t_i) - \widetilde{f}_m(t_1))$  ( $2 \leq i \leq s$ ). There exists an  $(\ell + m\ell) \times (\ell + m\ell)$  regular matrix  $Q_2$  such that

$$Q_1 M_2 Q_2 = \left( \begin{array}{c|cccc} E_\ell & & & & 0 \\ \hline 0 & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_2)} & & & 0 \\ & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_2)} & & \\ & 0 & & \ddots & \\ & & & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_2)} \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_s)} & & & 0 \\ & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_s)} & & \\ & 0 & & \ddots & \\ & & & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_s)} \end{array} \right)_{t=z},$$

By  $s - 1 \leq s_f - 1$  and the definition of  $s_f$ , it follows that

$$\dim \sum_{i=2}^s \mathbb{R} \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_i)} = s - 1,$$

where  $t = z$ . Thus, by the construction of the matrix  $Q_1 M_2 Q_2$  and by  $s - 1 \leq m$ , we have that the rank of the matrix  $Q_1 M_2 Q_2$  is  $\ell s$ . Hence, the rank of the matrix  $M_2$  must be  $\ell s$ . Therefore, we have (\*\*). Thus, there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , for any  $s$  ( $2 \leq s \leq s_f$ ), the mapping  $(F_\pi \circ f)^{(s)} : N^{(s)} \rightarrow (\mathbb{R}^\ell)^s$  is transverse to the submanifold  $\Delta_s$ .

Moreover, suppose that the mapping  $F_\pi$  satisfies that  $|F_\pi^{-1}(y)| \leq s_f$  for any  $y \in \mathbb{R}^\ell$ . Since  $f : N \rightarrow \mathbb{R}^m$  is injective, it follows that  $|(F_\pi \circ f)^{-1}(y)| \leq s_f$  for any  $y \in \mathbb{R}^\ell$ . Hence, it follows that for any positive integer  $s$  ( $s_f + 1 \leq s$ ),  $(F_\pi \circ f)^{(s)}(N^{(s)}) \cap \Delta_s = \emptyset$ . Namely, for any positive integer  $s$  ( $s_f + 1 \leq s$ ), the mapping  $(F_\pi \circ f)^{(s)}$  is transverse to  $\Delta_s$ . Thus, the mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is a mapping with normal crossings.  $\square$

## 5. APPLICATIONS OF THE TWO MAIN THEOREMS

In Subsection 5.1 (resp., Subsection 5.2), applications of Theorem 1 (resp., Theorem 2) are stated and proved. In Subsection 5.2, the applications obtained by combining Theorem 1 and Theorem 2 are also given.

### 5.1. Applications of Theorem 1. Set

$$\Sigma^k = \{j^1 g(0) \in J^1(n, \ell) \mid \text{corank } Jg(0) = k\},$$

where  $\text{corank } Jg(0) = \min\{n, \ell\} - \text{rank } Jg(0)$  and  $k = 1, \dots, \min\{n, \ell\}$ . Then,  $\Sigma^k$  is an  $\mathcal{A}^1$ -invariant submanifold of  $J^1(n, \ell)$ . Set

$$\Sigma^k(N, \mathbb{R}^\ell) = \bigcup_{\lambda \in \Lambda} \Phi_\lambda^{-1}(\varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times \Sigma^k),$$

where the mappings  $\Phi_\lambda$  and  $\varphi_\lambda$  are defined in Section 2. Then, the set  $\Sigma^k(N, \mathbb{R}^\ell)$  is a subfiber-bundle of  $J^1(N, \mathbb{R}^\ell)$  with the fiber  $\Sigma^k$  such that

$$\begin{aligned} \text{codim } \Sigma^k(N, \mathbb{R}^\ell) &= \dim J^1(N, \mathbb{R}^\ell) - \dim \Sigma^k(N, \mathbb{R}^\ell) \\ &= (n - v + k)(\ell - v + k), \end{aligned}$$

where  $v = \min\{n, \ell\}$ . (for details on  $\Sigma^k$  and  $\Sigma^k(N, \mathbb{R}^\ell)$ , see for example [3], pp.60–61).

As applications of Theorem 1, we have the following Proposition 1, Corollary 1, Corollary 2, Corollary 3 and Corollary 4.

**Proposition 1.** *Let  $N$  be a manifold of dimension  $n$ . Let  $f$  be an immersion of  $N$  into an open subspace  $U$  of  $\mathbb{R}^m$ . Let  $F : U \rightarrow \mathbb{R}^\ell$  be a mapping. Let  $k_0$  be the maximum integer satisfying  $(n - v + k_0)(\ell - v + k_0) \leq n$  ( $v = \min\{n, \ell\}$ ). Then, there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , the mapping  $j^1(F_\pi \circ f) : N \rightarrow J^1(N, \mathbb{R}^\ell)$  is transverse to the submanifold  $\Sigma^k(N, \mathbb{R}^\ell)$  for any positive integer  $k$  satisfying  $1 \leq k \leq v$ , and the mapping satisfies that  $j^1(F_\pi \circ f)(N) \cap \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$  for any positive integer  $k$  satisfying  $k_0 + 1 \leq k \leq v$ .*

*Proof.* By Theorem 1, for any positive integer  $k$  satisfying  $1 \leq k \leq v$ , there exists a subset  $\tilde{\Sigma}_k$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \tilde{\Sigma}_k$ , the mapping  $j^1(F_\pi \circ f) : N \rightarrow J^1(N, \mathbb{R}^\ell)$  is transverse to  $\Sigma^k(N, \mathbb{R}^\ell)$ . Set  $\Sigma = \bigcup_{k=1}^v \tilde{\Sigma}_k$ . Then, it is clearly seen that  $\Sigma$  is a subset of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$

with Lebesgue measure zero. Hence, it follows that there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , the mapping  $j^1(F_\pi \circ f) : N \rightarrow J^1(N, \mathbb{R}^\ell)$  is transverse to the submanifold  $\Sigma^k(N, \mathbb{R}^\ell)$  for any positive integer  $k$  satisfying  $1 \leq k \leq v$ ,

Now, we will show that the mapping  $j^1(F_\pi \circ f) : N \rightarrow J^1(N, \mathbb{R}^\ell)$  satisfies that  $j^1(F_\pi \circ f)(N) \cap \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$  for any positive integer  $k$  satisfying  $k_0 + 1 \leq k \leq v$ . Suppose that there exist a positive integer  $k$  ( $k_0 + 1 \leq k \leq v$ ) and a point  $q \in N$  such that  $j^1(F_\pi \circ f)(q) \in \Sigma^k(N, \mathbb{R}^\ell)$ . Since the mapping  $j^1(F_\pi \circ f) : N \rightarrow J^1(N, \mathbb{R}^\ell)$  is transverse to  $\Sigma^k(N, \mathbb{R}^\ell)$  at the point  $q$ , the following holds:

$$d(j^1(F_\pi \circ f))_q(T_q N) + T_{j^1(F_\pi \circ f)(q)} \Sigma^k(N, \mathbb{R}^\ell) = T_{j^1(F_\pi \circ f)(q)} J^1(N, \mathbb{R}^\ell).$$

Hence, we have

$$\begin{aligned} & \dim d(j^1(F_\pi \circ f))_q(T_q N) \\ & \geq \dim T_{j^1(F_\pi \circ f)(q)} J^1(N, \mathbb{R}^\ell) - \dim T_{j^1(F_\pi \circ f)(q)} \Sigma^k(N, \mathbb{R}^\ell) \\ & = \operatorname{codim} T_{j^1(F_\pi \circ f)(q)} \Sigma^k(N, \mathbb{R}^\ell). \end{aligned}$$

Thus, we have  $n \geq (n - v + k)(\ell - v + k)$ . Since the given integer  $k_0$  is the maximum integer satisfying  $n \geq (n - v + k_0)(\ell - v + k_0)$ , it follows that  $k \leq k_0$ . This contradicts the assumption  $k_0 + 1 \leq k$ .  $\square$

5.1.1. *Remark.* In Proposition 1, remark that  $k_0 \geq 0$ .

A mapping  $g : N \rightarrow \mathbb{R}$  is called a *Morse function* if all of the singularities of the mapping  $g$  are nondegenerate (for details on Morse function, see for example, [3], p.63). In the case of  $(n, \ell) = (n, 1)$ , we have the following.

**Corollary 1.** *Let  $N$  be a manifold of dimension  $n$ . Let  $f$  be an immersion of  $N$  into an open subspace  $U$  of  $\mathbb{R}^m$ . Let  $F : U \rightarrow \mathbb{R}$  be a mapping. Then, there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R})$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) - \Sigma$ , the mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}$  is a Morse function.*

*Proof.* By Proposition 1, there exists a subset  $\Sigma$  with Lebesgue measure zero of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R})$  such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) - \Sigma$ , the mapping  $j^1(F_\pi \circ f) : N \rightarrow J^1(N, \mathbb{R})$  is transverse to the submanifold  $\Sigma^1(N, \mathbb{R})$ . Hence, if  $q \in N$  is a singular point of the mapping  $F_\pi \circ f$ , then the point  $q$  is nondegenerate.  $\square$

For a given mapping  $g : N \rightarrow \mathbb{R}^{2n-1}$  ( $n \geq 2$ ), a singular point  $q \in N$  is called a *singular point of Whitney umbrella* if there exist two germs of diffeomorphisms  $H : (\mathbb{R}^\ell, g(q)) \rightarrow (\mathbb{R}^\ell, 0)$  and  $h : (N, q) \rightarrow (N, 0)$  such that  $H \circ g \circ h^{-1}(x_1, \dots, x_n) = (x_1^2, x_1 x_2, \dots, x_1 x_n, x_2, \dots, x_n)$ , where  $(x_1, \dots, x_n)$  is a local coordinate containing the point  $h(q) = 0 \in \mathbb{R}^n$ . In the case of  $(n, \ell) = (n, 2n - 1)$  ( $n \geq 2$ ), we have the following.

**Corollary 2.** *Let  $N$  be a manifold of dimension  $n$  ( $n \geq 2$ ). Let  $f$  be an immersion of  $N$  into an open subspace  $U$  of  $\mathbb{R}^m$ . Let  $F : U \rightarrow \mathbb{R}^{2n-1}$  be a mapping. Then, there exists a subset  $\Sigma$  with Lebesgue measure zero of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1})$  such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1}) - \Sigma$ , any singular point of the mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^{2n-1}$  is a singular point of Whitney umbrella.*

*Proof.* By for example, [3], p.179, we see that a point  $q \in N$  is a singular point of Whitney umbrella of the mapping  $F_\pi \circ f$  if  $j^1(F_\pi \circ f)(q) \in \Sigma^1(N, \mathbb{R}^{2n-1})$  and the mapping  $j^1(F_\pi \circ f)$  is transverse to the submanifold  $\Sigma^1(N, \mathbb{R}^{2n-1})$  at  $q$ . Set  $\ell = 2n - 1$  and  $v = n$  in Proposition 1. Then, it is clearly seen that we have  $k_0 = 1$  in Proposition 1. Hence, there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1})$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1}) - \Sigma$ , the mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^{2n-1}$  is transverse to  $\Sigma^k(N, \mathbb{R}^{2n-1})$  for any positive integer  $k$  satisfying  $1 \leq k \leq n$ , and the mapping satisfies that  $j^1(F_\pi \circ f)(N) \cap \Sigma^k(N, \mathbb{R}^{2n-1}) = \emptyset$  for any positive integer  $k$  satisfying  $2 \leq k \leq n$ . Thus, if a point  $q \in N$  is a singular point of the mapping  $F_\pi \circ f$ , then it follows that  $j^1(F_\pi \circ f)(q) \in \Sigma^1(N, \mathbb{R}^{2n-1})$  and  $j^1(F_\pi \circ f)$  is transverse to  $\Sigma^1(N, \mathbb{R}^{2n-1})$  at  $q$ .  $\square$

In the case of  $\ell \geq 2n$ , the immersed property of a given mapping  $f : N \rightarrow U$  is preserved by composing generic linearly perturbed mappings as follows:

**Corollary 3.** *Let  $N$  be a manifold of dimension  $n$ . Let  $f$  be an immersion of  $N$  into an open subspace  $U$  of  $\mathbb{R}^m$ . Let  $F : U \rightarrow \mathbb{R}^\ell$  be a mapping ( $\ell \geq 2n$ ). Then, there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , the mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is an immersion.*

*Proof.* It is clearly seen that the mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is an immersion if and only if it follows that  $j^1(F_\pi \circ f)(N) \cap \bigcup_{k=1}^n \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$ . Set  $v = n$  and  $\ell \geq 2n$  in Proposition 1. Then, it is clearly seen that  $k_0 \leq 0$ . By Remark 5.1.1, we get  $k_0 = 0$ . Hence, there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , the mapping  $j^1(F_\pi \circ f) : N \rightarrow J^1(N, \mathbb{R}^\ell)$  satisfies that  $j^1(F_\pi \circ f)(N) \cap \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$  for any positive integer  $k$  ( $1 \leq k \leq n$ ).  $\square$

For a give mapping  $g : N \rightarrow \mathbb{R}^\ell$ , the mapping  $g$  has corank at most  $k$  singular points if it follows that

$$\sup \{ \text{corank } dg_q \mid q \in N \} = k,$$

where  $\text{corank } dg_q = \min\{n, \ell\} - \text{rank } dg_q$ . By Proposition 1, we have the following corollary.

**Corollary 4.** *Let  $N$  be a manifold of dimension  $n$ . Let  $f$  be an immersion of  $N$  into an open subspace  $U$  of  $\mathbb{R}^m$ . Let  $F : U \rightarrow \mathbb{R}^\ell$  be a mapping. Let  $k_0$  be the maximum integer satisfying  $(n - v + k_0)(\ell - v + k_0) \leq n$  ( $v = \min\{n, \ell\}$ ). Then, there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , the mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  has corank at most  $k_0$  singular points.*

## 5.2. Applications of Theorem 2.

**Proposition 2.** *Let  $N$  be a manifold of dimension  $n$ . Let  $f$  be an injection of  $N$  into an open subspace  $U$  of  $\mathbb{R}^m$ . Let  $F : U \rightarrow \mathbb{R}^\ell$  be a mapping. If  $(s_f - 1)\ell > ns_f$ , then there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , the mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is a mapping with normal crossings satisfying  $(F_\pi \circ f)^{(s_f)}(N^{(s_f)}) \cap \Delta_{s_f} = \emptyset$ .*

*Proof.* By Theorem 2, there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , for any  $s$  ( $2 \leq s \leq s_f$ ), the mapping  $(F_\pi \circ f)^{(s)} : N^{(s)} \rightarrow (\mathbb{R}^\ell)^s$  is transverse to the submanifold  $\Delta_s$ . Hence, in order to show Proposition 2, it is sufficient to show that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , the mapping  $(F_\pi \circ f)^{(s_f)}$  satisfies that  $(F_\pi \circ f)^{(s_f)}(N^{(s_f)}) \cap \Delta_{s_f} = \emptyset$ .

Suppose that there exists an element  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$  such that there exists a point  $q \in N^{(s_f)}$  satisfying  $(F_\pi \circ f)^{(s_f)}(q) \in \Delta_{s_f}$ . Since  $(F_\pi \circ f)^{(s_f)}$  is transverse to  $\Delta_{s_f}$ , we have the following:

$$d((F_\pi \circ f)^{(s_f)})_q(T_q N^{(s_f)}) + T_{(F_\pi \circ f)^{(s_f)}(q)} \Delta_{s_f} = T_{(F_\pi \circ f)^{(s_f)}(q)} (\mathbb{R}^\ell)^{s_f}.$$

Hence, we have

$$\begin{aligned} & \dim d((F_\pi \circ f)^{(s_f)})_q(T_q N^{(s_f)}) \\ & \geq \dim T_{(F_\pi \circ f)^{(s_f)}(q)} (\mathbb{R}^\ell)^{s_f} - \dim T_{(F_\pi \circ f)^{(s_f)}(q)} \Delta_{s_f} \\ & = \text{codim } T_{(F_\pi \circ f)^{(s_f)}(q)} \Delta_{s_f}. \end{aligned}$$

Thus, we have  $ns_f \geq (s_f - 1)\ell$ . This contradicts the assumption  $(s_f - 1)\ell > ns_f$ .  $\square$

In the case of  $\ell > 2n$ , the injective property of a given mapping  $f : N \rightarrow U$  is preserved by composing generic linearly perturbed mappings as follows:

**Corollary 5.** *Let  $N$  be a manifold of dimension  $n$ . Let  $f$  be an injection of  $N$  into an open subspace  $U$  of  $\mathbb{R}^m$ . Let  $F : U \rightarrow \mathbb{R}^\ell$  be a mapping. If  $\ell > 2n$ , then there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , the mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is injective.*

*Proof.* Since  $s_f \geq 2$  and  $\ell > 2n$ , it is clearly seen that the dimension pair  $(n, \ell)$  satisfies the assumption  $(s_f - 1)\ell > ns_f$  of Proposition 2. Indeed, by  $\ell > 2n$ , it follows that  $(s_f - 1)\ell > 2n(s_f - 1)$ . By  $s_f \geq 2$ , we get  $2n(s_f - 1) \geq ns_f$ .

Hence, by Proposition 2, there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , the mapping  $(F_\pi \circ f)^{(2)} : N^{(2)} \rightarrow (\mathbb{R}^\ell)^2$  is transverse to  $\Delta_2$ . In order to show Corollary 5, it is sufficient to show that the mapping  $(F_\pi \circ f)^{(2)}$  satisfies that  $(F_\pi \circ f)^{(2)}(N^{(2)}) \cap \Delta_2 = \emptyset$ .

Suppose that there exists a point  $q \in N^{(2)}$  such that  $(F_\pi \circ f)^{(2)}(q) \in \Delta_2$ . Then, we have the following:

$$d((F_\pi \circ f)^{(2)})_q(T_q N^{(2)}) + T_{(F_\pi \circ f)^{(2)}(q)} \Delta_2 = T_{(F_\pi \circ f)^{(2)}(q)} (\mathbb{R}^\ell)^2.$$

Hence, we have

$$\begin{aligned} & \dim d((F_\pi \circ f)^{(2)})_q(T_q N^{(2)}) \\ & \geq \dim T_{(F_\pi \circ f)^{(2)}(q)} (\mathbb{R}^\ell)^2 - \dim T_{(F_\pi \circ f)^{(2)}(q)} \Delta_2 \\ & = \text{codim } T_{(F_\pi \circ f)^{(2)}(q)} \Delta_2. \end{aligned}$$

Thus, we have  $2n \geq \ell$ . This contradicts the assumption  $\ell > 2n$ .  $\square$

By combining Corollary 3 and Corollary 5, we have the following.

**Corollary 6.** *Let  $N$  be a manifold of dimension  $n$ . Let  $f$  be an injective immersion of  $N$  into an open subspace  $U$  of  $\mathbb{R}^m$ . Let  $F : U \rightarrow \mathbb{R}^\ell$  be a mapping. If  $\ell > 2n$ , then there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , the mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is an injective immersion.*

5.2.1. *Remark.* In Corollary 6, suppose that the mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is proper. Then, the injective immersion of  $F_\pi \circ f$  implies the embedding of it. (see [3], p.11). Thus, we get the following.

**Corollary 7.** *Let  $N$  be a compact manifold of dimension  $n$ . Let  $f$  be an embedding of  $N$  into an open subspace  $U$  of  $\mathbb{R}^m$ . Let  $F : U \rightarrow \mathbb{R}^\ell$  be a mapping. If  $\ell > 2n$ , then there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , the mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is an embedding.*

## 6. FURTHER APPLICATIONS

**6.1. Introduction of generalized distance-squared mappings.** Let  $i$  and  $j$  be positive integers, and let  $p_i = (p_{i1}, p_{i2}, \dots, p_{im})$  ( $1 \leq i \leq \ell$ ) (resp.,  $A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$ ) be a point of  $\mathbb{R}^m$  (resp., an  $\ell \times m$  matrix with non-zero entries). Set  $p = (p_1, p_2, \dots, p_\ell) \in (\mathbb{R}^m)^\ell$ . Let  $G_{(p,A)} : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$  be the mapping defined by

$$G_{(p,A)}(x) = \left( \sum_{j=1}^m a_{1j}(x_j - p_{1j})^2, \sum_{j=1}^m a_{2j}(x_j - p_{2j})^2, \dots, \sum_{j=1}^m a_{\ell j}(x_j - p_{\ell j})^2 \right),$$

where  $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$ . The mapping  $G_{(p,A)}$  is called a *generalized distance-squared mapping*, and the  $\ell$ -tuple of points  $p = (p_1, \dots, p_\ell) \in (\mathbb{R}^m)^\ell$  is called the *central point* of the generalized distance-squared mapping  $G_{(p,A)}$ . A *distance-squared mapping*  $D_p$  (resp., *Lorentzian distance-squared mapping*  $L_p$ ) is the mapping  $G_{(p,A)}$  satisfying that each entry of  $A$  is 1 (resp.,  $a_{i1} = -1$  and  $a_{ij} = 1$  ( $j \neq 1$ )).

In [5] (resp., [6]), a classification result on distance-squared mappings  $D_p$  (resp., Lorentzian distance-squared mappings  $L_p$ ) is given.

In [9], a classification result on generalized distance-squared mappings of the plane into the plane is given. If the rank of  $A$  is two, a generalized distance-squared mapping having a generic central point is a mapping of which any singular point is a fold point except one cusp point. The singular set is a rectangular hyperbola. If the rank of  $A$  is one, a generalized distance-squared mapping having a generic central point is  $\mathcal{A}$ -equivalent to the normal form of fold singularity  $(x_1, x_2) \mapsto (x_1, x_2^2)$ .

In [7], a classification result on generalized distance-squared mappings of  $\mathbb{R}^{m+1}$  into  $\mathbb{R}^{2m+1}$  is given. If the rank of  $A$  is  $m+1$ , a generalized distance-squared mapping having a generic central point is  $\mathcal{A}$ -equivalent to the normal form of Whitney umbrella  $(x_1, \dots, x_{m+1}) \mapsto (x_1^2, x_1x_2, \dots, x_1x_{m+1}, x_2, \dots, x_{m+1})$ . If the rank of  $A$  is less than  $m+1$ , a generalized distance-squared mapping having a generic central point is  $\mathcal{A}$ -equivalent to the inclusion  $(x_1, \dots, x_{m+1}) \mapsto (x_1, \dots, x_{m+1}, 0, \dots, 0)$ .

Namely, in [5], [6], [7] and [9], the properties of generic generalized distance-squared mappings are investigated. Hence, it is natural to investigate the properties of compositions with generic generalized distance-squared mappings.

We have another original motivation. Height functions and distance-squared functions have been investigated in detail so far, and they are a useful tool in the applications of singularity theory to differential geometry (for instance, see [2]). The

mapping in which each component is a height function is nothing but a projection. In [10], compositions of generic projections and embeddings are investigated.

On the other hand, the mapping in which each component is a distance-squared function is a distance-squared mapping. In addition, the notion of generalized distance-squared mapping is an extension of the distance-squared mappings. Therefore, it is again natural to investigate compositions with generic generalized distance-squared mappings.

## 6.2. Applications of Theorem 1 to $G_{(p,A)} : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$ .

**Proposition 3.** *Let  $N$  be a manifold of dimension  $n$ . Let  $f : N \rightarrow \mathbb{R}^m$  be an immersion. If  $X$  is an  $\mathcal{A}^1$ -invariant submanifold of  $J^1(n, \ell)$ , then there exists a subset  $\Sigma$  of  $(\mathbb{R}^m)^\ell$  with Lebesgue measure zero such that for any  $p \in (\mathbb{R}^m)^\ell - \Sigma$ , the mapping  $j^1(G_{(p,A)} \circ f) : N \rightarrow J^1(N, \mathbb{R}^\ell)$  is transverse to the submanifold  $X(N, \mathbb{R}^\ell)$ .*

*Proof.* Let  $H : \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell$  be the diffeomorphism of the target for deleting constant terms. The composition  $H \circ G_{(p,A)} : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$  is as follows:

$$H \circ G_{(p,A)}(x) = \left( \sum_{j=1}^m a_{1j}x_j^2 - 2 \sum_{j=1}^m a_{1j}p_{1j}x_j, \dots, \sum_{j=1}^m a_{\ell j}x_j^2 - 2 \sum_{j=1}^m a_{\ell j}p_{\ell j}x_j \right),$$

where  $x = (x_1, \dots, x_m)$ .

Let  $\psi : (\mathbb{R}^m)^\ell \rightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  be the mapping defined by

$$\psi(p_{11}, p_{12}, \dots, p_{\ell m}) = -2(a_{11}p_{11}, a_{12}p_{12}, \dots, a_{\ell m}p_{\ell m}).$$

Remark that there exists the natural identification  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$ . Since  $a_{ij} \neq 0$  for any  $i, j$  ( $1 \leq i \leq \ell$ ,  $1 \leq j \leq m$ ), it is clearly seen that  $\psi$  is a  $C^\infty$  diffeomorphism.

Set  $F_i(x) = \sum_{j=1}^m a_{ij}x_j^2$  ( $1 \leq i \leq \ell$ ) and  $F = (F_1, \dots, F_\ell)$ . By Theorem 1, there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , the mapping  $j^1(F_\pi \circ f) : N \rightarrow J^1(N, \mathbb{R}^\ell)$  is transverse to  $X(N, \mathbb{R}^\ell)$ . Since  $\psi^{-1} : \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \rightarrow (\mathbb{R}^m)^\ell$  is a  $C^\infty$  mapping,  $\psi^{-1}(\Sigma)$  is a subset of  $(\mathbb{R}^m)^\ell$  with Lebesgue measure zero. For any  $p \in (\mathbb{R}^m)^\ell - \psi^{-1}(\Sigma)$ , we have  $\psi(p) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ . Hence, for any  $p \in (\mathbb{R}^m)^\ell - \psi^{-1}(\Sigma)$ , the mapping  $j^1(H \circ G_{(p,A)} \circ f) : N \rightarrow J^1(N, \mathbb{R}^\ell)$  is transverse to  $X(N, \mathbb{R}^\ell)$ . Then, since  $H : \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell$  is a diffeomorphism, the mapping  $j^1(G_{(p,A)} \circ f) : N \rightarrow J^1(N, \mathbb{R}^\ell)$  is transverse to  $X(N, \mathbb{R}^\ell)$ .  $\square$

6.2.1. *Remark.* As applications of Proposition 3, regarding generalized distance-squared mappings, we get analogies of Proposition 1, Corollary 1, Corollary 2, Corollary 3 and Corollary 4.

## 6.3. Applications of Theorem 2 to $G_{(p,A)} : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$ .

**Proposition 4.** *Let  $N$  be a manifold of dimension  $n$ . Let  $f : N \rightarrow \mathbb{R}^m$  be an injection. Then, there exists a subset  $\Sigma$  of  $(\mathbb{R}^m)^\ell$  with Lebesgue measure zero such that for any  $p \in (\mathbb{R}^m)^\ell - \Sigma$ , for any  $s$  ( $2 \leq s \leq s_f$ ), the mapping  $(G_{(p,A)} \circ f)^{(s)} : N^{(s)} \rightarrow (\mathbb{R}^\ell)^s$  is transverse to the submanifold  $\Delta_s$ . Moreover, if the mapping  $G_{(p,A)}$  satisfies that  $|G_{(p,A)}^{-1}(y)| \leq s_f$  for any  $y \in \mathbb{R}^\ell$ , then the mapping  $G_{(p,A)} \circ f : N \rightarrow \mathbb{R}^\ell$  is a mapping with normal crossings, where  $|X|$  is the number of elements of the set  $X$ .*



*Proof.* The method of the proof of this proposition is the same as the method of the proof of Proposition 3. Let  $H : \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell$  be the diffeomorphism of the target for deleting constant terms. The composition  $H \circ G_{(p,A)} : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$  is as follows:

$$H \circ G_{(p,A)}(x) = \left( \sum_{j=1}^m a_{1j}x_j^2 - 2 \sum_{j=1}^m a_{1j}p_{1j}x_j, \dots, \sum_{j=1}^m a_{\ell j}x_j^2 - 2 \sum_{j=1}^m a_{\ell j}p_{\ell j}x_j \right),$$

where  $x = (x_1, \dots, x_m)$ .

Let  $\psi : (\mathbb{R}^m)^\ell \rightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  be the mapping defined by

$$\psi(p_{11}, p_{12}, \dots, p_{\ell m}) = -2(a_{11}p_{11}, a_{12}p_{12}, \dots, a_{\ell m}p_{\ell m}).$$

Remark that there exists the natural identification  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$ . Since  $a_{ij} \neq 0$  for any  $i, j$  ( $1 \leq i \leq \ell$ ,  $1 \leq j \leq m$ ), it is clearly seen that  $\psi$  is a  $C^\infty$  diffeomorphism.

Set  $F_i(x) = \sum_{j=1}^m a_{ij}x_j^2$  ( $1 \leq i \leq \ell$ ) and  $F = (F_1, \dots, F_\ell)$ . By Theorem 2, there exists a subset  $\Sigma$  of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  with Lebesgue measure zero such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ , for any  $s$  ( $2 \leq s \leq s_f$ ), the mapping  $(F_\pi \circ f)^{(s)} : N^{(s)} \rightarrow (\mathbb{R}^\ell)^s$  is transverse to  $\Delta_s$ . Moreover, if the mapping  $F_\pi$  satisfies that  $|F_\pi^{-1}(y)| \leq s_f$  for any  $y \in \mathbb{R}^\ell$ , then the mapping  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is a mapping with normal crossings. Since  $\psi^{-1} : \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \rightarrow (\mathbb{R}^m)^\ell$  is a  $C^\infty$  mapping,  $\psi^{-1}(\Sigma)$  is a subset of  $(\mathbb{R}^m)^\ell$  with Lebesgue measure zero. For any  $p \in (\mathbb{R}^m)^\ell - \psi^{-1}(\Sigma)$ , we have  $\psi(p) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ . Hence, for any  $p \in (\mathbb{R}^m)^\ell - \psi^{-1}(\Sigma)$ , for any  $s$  ( $2 \leq s \leq s_f$ ), the mapping  $(H \circ G_{(p,A)} \circ f)^{(s)} : N^{(s)} \rightarrow (\mathbb{R}^\ell)^s$  is transverse to  $\Delta_s$ . Moreover, if the mapping  $H \circ G_{(p,A)}$  satisfies that  $|(H \circ G_{(p,A)})^{-1}(y)| \leq s_f$  for any  $y \in \mathbb{R}^\ell$ , then the mapping  $H \circ G_{(p,A)} \circ f : N \rightarrow \mathbb{R}^\ell$  is a mapping with normal crossings. Then, since  $H : \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell$  is a diffeomorphism, the assertion of this proposition holds.  $\square$

**6.3.1. Remark.** As applications of Proposition 4, regarding generalized distance-squared mappings, we get analogies of Proposition 2, Corollary 5, Corollary 6 and Corollary 7.

As the special case of the classification result of distance squared mappings  $D_p$  (resp., Lorentzian distance-squared mappings  $L_p$ ) in [5] (resp., [6]), we have Lemma 6.1 (resp., Lemma 6.2).

**Lemma 6.1** ([5]). *Let  $D_p : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$  be a distance-squared mapping. Then, the following hold:*

- (1) *In the case of  $m = \ell = 1$ , for any  $p \in \mathbb{R}$ , the mapping  $D_p : \mathbb{R} \rightarrow \mathbb{R}$  is  $\mathcal{A}$ -equivalent to  $x \mapsto x^2$ .*
- (2) *In the case of  $2 \leq m = \ell$ , there exists a subset  $\Sigma$  of  $(\mathbb{R}^m)^m$  with Lebesgue measure zero such that for any  $p \in (\mathbb{R}^m)^m - \Sigma$ , the mapping  $D_p : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is  $\mathcal{A}$ -equivalent to the normal form of definite fold mappings  $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_{m-1}, x_m^2)$ .*
- (3) *In the case of  $1 \leq m < \ell$ , there exists a subset  $\Sigma$  of  $(\mathbb{R}^m)^\ell$  with Lebesgue measure zero such that for any  $p \in (\mathbb{R}^m)^\ell - \Sigma$ , the mapping  $D_p : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$  is  $\mathcal{A}$ -equivalent to the inclusion  $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$ .*

**Lemma 6.2** ([6]). *Let  $L_p : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$  be a Lorentzian distance-squared mapping. Then, the following hold:*

- (1) In the case of  $m = \ell = 1$ , for any  $p \in \mathbb{R}$ , the mapping  $L_p : \mathbb{R} \rightarrow \mathbb{R}$  is  $\mathcal{A}$ -equivalent to  $x \mapsto x^2$ .
- (2) In the case of  $2 \leq m = \ell$ , there exists a subset  $\Sigma$  of  $(\mathbb{R}^m)^m$  with Lebesgue measure zero such that for any  $p \in (\mathbb{R}^m)^m - \Sigma$ , the mapping  $L_p : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is  $\mathcal{A}$ -equivalent to the normal form of definite fold mappings  $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_{m-1}, x_m^2)$ .
- (3) In the case of  $1 \leq m < \ell$ , there exists a subset  $\Sigma$  of  $(\mathbb{R}^m)^\ell$  with Lebesgue measure zero such that for any  $p \in (\mathbb{R}^m)^\ell - \Sigma$ , the mapping  $L_p : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$  is  $\mathcal{A}$ -equivalent to the inclusion  $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$ .

Notice that in the case of  $m < \ell$ , since  $D_p : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$  and  $L_p : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$  having a generic central point are  $\mathcal{A}$ -equivalent to the inclusion, their cases are trivial. Thus, we will consider the case of  $m = \ell$ .

**Proposition 5.** . Let  $N$  be a manifold of dimension  $n$ . Let  $f : N \rightarrow \mathbb{R}^m$  be an injection. Let  $D_p : \mathbb{R}^m \rightarrow \mathbb{R}^m$  (resp.,  $L_p : \mathbb{R}^m \rightarrow \mathbb{R}^m$ ) be a distance-squared mapping (resp., Lorentzian distance-squared mapping). Then, the following hold:

- (1) There exists a subset  $\Sigma$  of  $(\mathbb{R}^m)^m$  with Lebesgue measure zero such that for any  $p \in (\mathbb{R}^m)^m - \Sigma$ , the mapping  $D_p \circ f : N \rightarrow \mathbb{R}^m$  is a mapping with normal crossings.
- (2) There exists a subset  $\Sigma$  of  $(\mathbb{R}^m)^m$  with Lebesgue measure zero such that for any  $p \in (\mathbb{R}^m)^m - \Sigma$ , the mapping  $L_p \circ f : N \rightarrow \mathbb{R}^m$  is a mapping with normal crossings.

*Proof.* The proofs of (1) and (2) of this proposition are almost the same. Thus, it is sufficient to show (1). By the assertions (1) and (2) of Lemma 6.1, there exists a subset  $\Sigma_1$  of  $(\mathbb{R}^m)^m$  with Lebesgue measure zero such that for any  $p \in (\mathbb{R}^m)^m - \Sigma_1$ , the mapping  $D_p : \mathbb{R}^m \rightarrow \mathbb{R}^m$  satisfies that  $|D_p(y)| \leq 2$  for any  $y \in \mathbb{R}^\ell$ .

On the other hand, by Proposition 4, there exists a subset  $\Sigma_2$  of  $(\mathbb{R}^m)^m$  with Lebesgue measure zero such that for any  $p \in (\mathbb{R}^m)^m - \Sigma_2$ , if the mapping  $D_p$  satisfies that  $|D_p^{-1}(y)| \leq s_f$  for any  $y \in \mathbb{R}^m$ , then the mapping  $D_p \circ f : N \rightarrow \mathbb{R}^m$  is a mapping with normal crossings. Set  $\Sigma = \Sigma_1 \cup \Sigma_2$ . It is clearly seen that  $\Sigma$  is a subset of  $(\mathbb{R}^m)^m$  with Lebesgue measure zero. Then, for any  $p \in (\mathbb{R}^m)^m - \Sigma$ , the mapping  $D_p \circ f : N \rightarrow \mathbb{R}^m$  is a mapping with normal crossings.  $\square$

By combining Proposition 5 and the analogy of Corollary 3 in Remark 6.2.1, we have the following.

**Corollary 8.** Let  $N$  be a manifold of dimension  $n$ . Let  $f : N \rightarrow \mathbb{R}^m$  be an injective immersion ( $2n \leq m$ ). Let  $D_p : \mathbb{R}^m \rightarrow \mathbb{R}^m$  (resp.,  $L_p : \mathbb{R}^m \rightarrow \mathbb{R}^m$ ) be a distance-squared mapping (resp., Lorentzian distance-squared mapping). Then, the following hold:

- (1) There exists a subset  $\Sigma$  of  $(\mathbb{R}^m)^m$  with Lebesgue measure zero such that for any  $p \in (\mathbb{R}^m)^m - \Sigma$ , the mapping  $D_p \circ f : N \rightarrow \mathbb{R}^m$  is an immersion with normal crossings.
- (2) There exists a subset  $\Sigma$  of  $(\mathbb{R}^m)^m$  with Lebesgue measure zero such that for any  $p \in (\mathbb{R}^m)^m - \Sigma$ , the mapping  $L_p \circ f : N \rightarrow \mathbb{R}^m$  is an immersion with normal crossings.

6.3.2. *Remark.* In Corollary 8, if the mapping  $D_p \circ f : N \rightarrow \mathbb{R}^{2n}$  (resp.,  $L_p \circ f : N \rightarrow \mathbb{R}^{2n}$ ) is proper, then the immersion with normal crossings of  $D_p \circ f : N \rightarrow \mathbb{R}^{2n}$  (resp.,  $L_p \circ f : N \rightarrow \mathbb{R}^{2n}$ ) implies the stability of it (see [3], p.86). Thus, we get the following.

**Corollary 9.** *Let  $N$  be a compact manifold of dimension  $n$ . Let  $f : N \rightarrow \mathbb{R}^{2n}$  be an embedding. Let  $D_p : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  (resp.,  $L_p : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ ) be a distance-squared mapping (resp., Lorentzian distance-squared mapping). Then, the following hold:*

- (1) *There exists a subset  $\Sigma$  of  $(\mathbb{R}^{2n})^{2n}$  with Lebesgue measure zero such that for any  $p \in (\mathbb{R}^{2n})^{2n} - \Sigma$ , the mapping  $D_p \circ f : N \rightarrow \mathbb{R}^{2n}$  is stable.*
- (2) *There exists a subset  $\Sigma$  of  $(\mathbb{R}^{2n})^{2n}$  with Lebesgue measure zero such that for any  $p \in (\mathbb{R}^{2n})^{2n} - \Sigma$ , the mapping  $L_p \circ f : N \rightarrow \mathbb{R}^{2n}$  is stable.*

We will compare Corollary 7 and Corollary 9 from the viewpoint of stability. Remark that the dimension of the target space in Corollary 9 is less than the dimension of the target space in Corollary 7.

## 7. APPENDIX

In this section, the main theorem in [4] and the main theorem in [10] are introduced. Remark that this section is prepared only for Remark 1.1 in Section 1. Hence, this section is not necessary except for Remark 1.1. For the introduction of the main theorem in [4] and the main theorem in [10], we prepare some notions.

Let  $N$  and  $P$  be manifolds. Let  ${}_s J^r(N, P)$  be the space consisting of elements  $(j^r g(q_1), \dots, j^r g(q_s)) \in J^r(N, P)^s$  satisfying  $(q_1, \dots, q_s) \in N^{(s)}$ . Since  $N^{(s)}$  is an open submanifold of  $N^s$ , the space  ${}_s J^r(N, P)$  is also an open submanifold of  $J^r(N, P)^s$ . For a given mapping  $g : N \rightarrow P$ , the mapping  ${}_s j^r g : N^{(s)} \rightarrow {}_s J^r(N, P)$  is defined by  $(q_1, \dots, q_s) \mapsto (j^r g(q_1), \dots, j^r g(q_s))$ .

Let  $W$  be a submanifold of  ${}_s J^r(N, P)$ . A mapping  $g : N \rightarrow P$  will be said to be *transverse with respect to  $W$*  if  ${}_s j^r g : N^{(s)} \rightarrow {}_s J^r(N, P)$  is transverse to  $W$ .

Following Mather ([10]), we can partition  $P^s$  as follows. Given any partition  $\pi$  of  $\{1, \dots, s\}$ , let  $P^\pi$  denote the set of  $s$ -tuples  $(y_1, \dots, y_s) \in P^s$  such that  $y_i = y_j$  if and only if two positive integers  $i$  and  $j$  are in the same member of the partition  $\pi$ .

Let  $\text{Diff } N$  denote the group of diffeomorphisms of  $N$ . There is a natural action of  $\text{Diff } N \times \text{Diff } P$  on  ${}_s J^r(N, P)$  such that for a mapping  $g : N \rightarrow P$ , the equality  $(h, H) \cdot {}_s j^r g(q) = {}_s j^r (H \circ g \circ h^{-1})(h(q))$  holds. A subset  $W$  of  ${}_s J^r(N, P)$  is called *invariant* if it is invariant under this action.

We recall the following identification (\*\*\*) from [10]. Let  $q = (q_1, \dots, q_s) \in N^{(s)}$ , let  $g : U \rightarrow P$  be a mapping defined in a neighborhood  $U$  of  $\{q_1, \dots, q_s\}$  in  $N$ , and let  $z = {}_s j^r g(q)$ ,  $q' = (g(q_1), \dots, g(q_s))$ . Let  ${}_s J^r(N, P)_q$  and  ${}_s J^r(N, P)_{q, q'}$  denote the fibers of  ${}_s J^r(N, P)$  over  $q$  and over  $(q, q')$  respectively. Let  $J^r(N)_q$  denote the  $\mathbb{R}$ -algebra of  $r$ -jets at  $q$  of functions on  $N$ . Namely,

$$J^r(N)_q = {}_s J^r(N, \mathbb{R})_q.$$

Set  $g^* TP = \bigcup_{q \in N} T_{g(q)} P$ , where  $TP$  is the tangent bundle of  $P$ . Let  $J^r(g^* TP)_q$  denote the  $J^r(N)_q$ -module of  $r$ -jets at  $q$  of sections of the bundle  $g^* TP$ . Let  $\mathfrak{m}_q$  be the ideal in  $J^r(N)_q$  consisting of jets of functions which vanish at  $q$ . Namely,

$$\mathfrak{m}_q = \{{}_s j^r h(q) \in {}_s J^r(N, \mathbb{R})_q \mid h(q_1) = \dots = h(q_s) = 0\}.$$

Let  $\mathfrak{m}_q J^r(g^*TP)_q$  be the set consisting of finite sums of the product of an element of  $\mathfrak{m}_q$  and an element of  $J^r(g^*TP)_q$ . Namely, we get

$$\mathfrak{m}_q J^r(g^*TP)_q = J^r(g^*TP)_q \cap \{ {}_s j^r \xi(q) \in {}_s J^r(N, TP)_q \mid \xi(q_1) = \cdots = \xi(q_s) = 0 \}.$$

Then, it is easily seen that the following canonical identification of  $\mathbb{R}$  vector spaces  $(***)$  holds.

$$(***) \quad T({}_s J^r(N, P)_{q, q'})_z = \mathfrak{m}_q J^r(g^*TP)_q.$$

Let  $W$  be a non-empty submanifold of  ${}_s J^r(N, P)$ . Choose  $q = (q_1, \dots, q_s) \in N^{(s)}$  and  $g : N \rightarrow P$ , and let  $z = {}_s j^r g(q)$  and  $q' = (g(q_1), \dots, g(q_s))$ . Suppose that the choice is made so that  $z \in W$ . Let  $W_{q, q'}$  denote the fiber of  $W$  over  $(q, q')$ .

Then, under the identification  $(***)$ , the tangent space  $T(W_{q, q'})_z$  can be identified with a vector subspace of  $\mathfrak{m}_q J^r(g^*TP)_q$ . We denote this vector subspace by  $E(g, q, W)$ .

**Definition 3.** The submanifold  $W$  is called *modular* if conditions  $(\alpha)$  and  $(\beta)$  below are satisfied:

- $(\alpha)$   $W$  is an invariant submanifold of  ${}_s J^r(N, P)$ , and lies over  $P^\pi$  for some partition  $\pi$  of  $\{1, \dots, s\}$ .
- $(\beta)$  For any  $q \in N^{(s)}$  and any mapping  $g : N \rightarrow P$  such that  ${}_s j^r g(q) \in W$ , the subspace  $E(g, q, W)$  is a  $J^r(N)_q$ -submodule.

Now, suppose that  $P = \mathbb{R}^\ell$ . The main theorem in [10] is the following.

**Theorem 3** ([10]). *Let  $N$  be a manifold of dimension  $n$ . Let  $f$  be an embedding of  $N$  into  $\mathbb{R}^m$ . If  $W$  is a modular submanifold of  ${}_s J^r(N, \mathbb{R}^\ell)$  and  $m > \ell$ , then there exists a subset  $\Sigma$  with Lebesgue measure zero of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ ,  $\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is transverse with respect to  $W$ .*

Then, the main theorem in [4] is the following.

**Theorem 4** ([4]). *Let  $N$  be a manifold of dimension  $n$ . Let  $f$  be an embedding of  $N$  into an open subspace  $U$  of  $\mathbb{R}^m$ . Let  $F : U \rightarrow \mathbb{R}^\ell$  be a mapping. If  $W$  is a modular submanifold of  ${}_s J^r(N, \mathbb{R}^\ell)$ , then there exists a subset  $\Sigma$  with Lebesgue measure zero of  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$  such that for any  $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ ,  $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$  is transverse with respect to  $W$ .*

The assertion (6) in Section 1, Corollary 7 in Section 5 and Corollary 9 in Section 6 are obtained as corollaries of Theorem 1 and Theorem 2. However, as described in Remark 1.1 in Section 1, remark that they are also corollaries obtained by using Theorem 4.

#### ACKNOWLEDGEMENTS

The author is grateful to Takashi Nishimura for his kind advices. The author is supported by JSPS KAKENHI Grant Number 16J06911.

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